DISCOVERY

Extending a hands-on task

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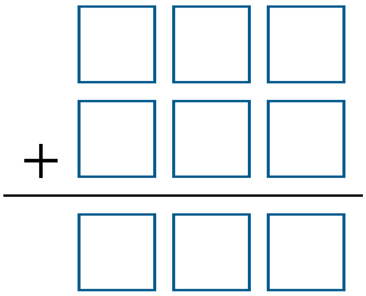
Consider the following hands-on task called Number Tiles, which is Task 43 in the collection constructed for the Mathematics Task Centre Project, and available at www.blackdouglas.com.au or www.curriculum.edu.au.
The task has nine discs with the digits 1 to 9 on them, and the following chart.



Place the discs in appropriate squares on the chart so that they make a correct addition sum.

There are many solutions to this problem, so with some trials you can soon obtain one of them. An interesting extension is to find out how many different solutions there are. To do this, let us break the problem up into three separate situations.

The first is when there is no 1 carried over from any column to the next. The second is when there is 1 carried over from two columns to the next (which must be from column 1 to column 2 AND from column 2 to column 3). The third is when there is 1 carried over from one column only to the next. (Column 1 to column 2 OR column 2 to column 3).

Next, consider the digits 1 to 9. There are five odd digits and four even digits, that is,

(ODD, EVEN) = (5,4).

When there is no carry over of 1 from the previous column, then the column must contain

EVEN + EVEN = EVEN
or ODD + ODD = EVEN
or ODD + EVEN = ODD

Hence, for this column

(ODD, EVEN) = (0,3) or (2,1)

For three columns with no carry over of 1, we need

(ODD, EVEN) = (0,9) or (2,7) or (4,5) or (6,3)

However, none of these are (5,4) and so no solution is possible without a carry over of 1.

When there is a carry over of 1 from the previous column then the column must contain

EVEN + EVEN + 1 = ODD
or ODD + ODD + 1 = ODD
or ODD + EVEN + 1 = EVEN

Hence, for each column with a carry over, we have

(ODD, EVEN) = (1,2) or (3,0)

If two columns involve a carry over of 1 and one column does not, then the total for three columns is

(ODD, EVEN) = (2,7) or (4,5) or (6,3) or (8,1)

Again, none of these are (5,4) and so no solution is possible with two columns involving a carry over 1.

Finally, we consider only one column having a carry over of 1, then the total for three columns is

(ODD, EVEN) = (1,8) or (3,6) or (5,4) or (7,2)

At last we have (5,4), indicating that solutions may be possible but must involve either

(a) ODD + ODD + 1 = ODD
ODD + EVEN = ODD
EVEN + EVEN = EVEN

(b) EVEN + EVEN + 1 = ODD
ODD + EVEN = ODD
ODD + ODD = EVEN

or

(c) ODD + EVEN + 1 = EVEN
ODD + EVEN = ODD
ODD + ODD = EVEN

Case (a)

For EVEN + EVEN = EVEN in one column and no carry over of 1 to the next column, we can only have

$2 + 4 = 6$ or $2 + 6 = 8$

Now these can only be in columns 1 (the units column) or 3 (the hundreds column). For $2 + 4 = 6$ in column 3, say, then the remaining even digit 8 must be in column 1. The odd digits that can be added to this to give a carry over in column 2 are 3, 5, 7 or 9. It is soon seen that 3 and 7 cannot produce a solution, but 5 yields $215 + 478 = 693$ while 9 yields $218 + 439 = 657$.

For $2 + 6 = 8$ in column 3 then the remaining even digit is 4 in column 1. The only odd digits that can be added to give a carry over of 1 in column 2 are 7 and 9. These produce $234 + 657 = 891$ and $214 + 659 = 873$ respectively.

Placing $2 + 4 = 6$ or $2 + 6 = 8$ in column 1 yields four closely related solutions to the ones already obtained. The new solutions are $152 + 784 = 936$, $182 + 394 = 576$, $342 + 576 = 918$ and $142 + 596 = 738$.

For EVEN + EVEN = EVEN and a carry over of 1 to the next column we can only have

$4 + 8 = (1)2$ or $6 + 8 = (1)4$

Similar deductions about the remaining even digits yield

$314 + 658 = 972$	and	$143 + 586 = 729$
$134 + 658 = 792$		$341 + 586 = 927$
$216 + 378 = 594$		$162 + 783 = 945$
$216 + 738 = 954$		$162 + 387 = 549$

Case (b)

Since the carried over column in this case is

EVEN + EVEN + 1 = ODD

and the sum has to be <10 (because no more carried overs are allowed) then we have either

$2 + 4 + ① = 7$ or $2 + 6 + ① = 9$

Placing one of each of the remaining even numbers in the columns 1 and 3 respectively reveals no solutions for $2 + 4 + ① = 7$, but four solutions for $2 + 6 + ① = 9$, namely

$324 + 567 = 891$
$127 + 368 = 495$

and the closely related solutions

$243 + 675 = 918$
$271 + 683 = 954$

obtained by the cyclic permutation of columns 1, 2, 3 by one step.

Case (c)

Now the carried over column is

ODD + EVEN + 1 = EVEN

and once again the sum has to be <10, because no more carried overs are allowed. The six possibilities are

(c1) $1 + 2 + ① = 4$
(c2) $1 + 4 + ① = 6$
(c3) $1 + 6 + ① = 8$
(c4) $3 + 2 + ① = 6$
(c5) $3 + 4 + ① = 8$
(c6) $5 + 2 + ① = 8$

We consider only (c1) in detail and leave the remainder as similar exercises for your students. For (c1) the remaining two even digits 6 and 8 must be in separate columns of the addition.

Suppose (c1) is the final addition in column 3. Then if column 2 contains 6 it can be in the addends or the total. This yields no solution with 6 in the addend, but $173 + 295 = 468$ for 6 in the total. If column 2 contains 8 it yields $173 + 286 = 459$ for the addend, and no solution for the total. As before the units column can be cycled to the hundreds column yielding the closely related solutions

$317 + 529 = 846$ and $317 + 628 = 945$

Carrying out a similar procedure on (c2) yields

215 + 748 = 963 , 218 + 349 = 567,
152 + 487 = 639 , 182 + 493 = 675,

on (c3) yields

214 + 569 = 783 , 142 + 695 = 837,

on (c4) yields

125 + 739 = 864 , 251 + 397 = 648,
128 + 439 = 567 , 281 + 394 = 675,

on (c5) yields

235 + 746 = 981 , 352 + 467 = 819,

and finally, on (c6) yields

127 + 359 = 486 , 271 + 593 = 864,
124 + 659 = 783 , 241 + 596 = 837,
243 + 576 = 819 , 324 + 657 = 981

We have now completed all possibilities and written down 42 solutions. But each one of these can produce another seven solutions by exchanging the digits in rows 1 and 2, one at a time. For example, the fourth solution in case (b)

271 + 683 = 954

produces also

671 + 283 = 954
281 + 673 = 954
681 + 273 = 954
273 + 681 = 954
673 + 281 = 954
283 + 671 = 954
683 + 271 = 954

Of course, the latter four solutions are just the earlier four with row 1 completely swapped with row 2, but these are all acceptable alternative solutions to the hands-on problem posed. Hence we have discovered that there are 336 possible solutions to this problem, and each one of them involves carrying 1 over from one column only.

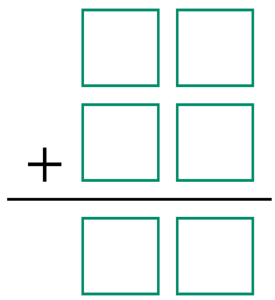
It is useful to note that the method of solving this problem, by looking at combinations of ODD and EVEN digits that were possible, greatly simplified the possibilities to be investigated. A systematic trial-and-error approach would involve an investigation of all the 9! (= 362 880) possible placements of the 9 digits in the 9 squares available to see if each addition was correct — a really daunting task unless a computer program can be written to do this. Such a program is available through the Maths 300 Software on the earlier website given but it does not help your problem-solving strategies.

Note also that dealing with the EVEN numbers first of all was a very good strategy to employ when trying to solve this problem analytically.

This task is rich in possibilities and directions. It should be used as a planned curriculum experience at several year levels to achieve the most from it. A wide range of students can try this problem, and then exit from it at different points feeling that they have gained some success in finding some of the solutions.

The first level of success is to find one solution, which usually takes some time to discover. The next is to find more than one, and then to start to see connections between numbers that work and numbers that do not work.

It is important initially to use a hands-on approach, and even nine pieces of paper with the numbers 1 to 9 on them will do. Perhaps your students can create other problems like this, for example, using any six of the digits from 1 to 9 to solve



Happy discoveries!